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Interpolation Formulas for Harmonic Functions

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It is known that a real-valued entire harmonic function u of exponential type less than π is uniquely determined by its values at the points n and $ne^{i\alpha}$, $n=0, \pm 1, \pm 2, ...$, unless α is a rational multiple of π. For $\alpha=\pi/2$, which belongs to the exceptional cases, Ching has proved that u is uniquely determined by its values at these points if u is in addition an odd function. In the present paper we shall extend this result to the case $\alpha=(2k+1)\pi/(2l)$, where k and $l\neq 0$ are arbitrary integers. Furthermore, we shall present formulas which allow a reconstruction of real-valued entire harmonic functions of exponential type π by their samples at the points n and $ne^{i\alpha}$, $n=0, \pm 1, \pm 2, ...$, when $\alpha=(2k+1)\pi/(2l)$ or when α/π is irrational and algebraic. © 1999 Academic Press

1. INTRODUCTION AND RESULTS

There exist numerous uniqueness theorems and interpolation formulas for entire harmonic functions in the plane (see, e.g., [3, 5–7, 10, 13, 14]). These investigations started with Boas [3, Theorem 1] who proved that an entire harmonic function of exponential type less than π is uniquely determined by its values at the lattice points n and $n+i$, $n=0, \pm 1, \pm 2, ...$. Besides, he obtained the following result [3, Theorem 2]; for an analogue in higher dimensions see [2].

Theorem A. *Let u be a real-valued entire harmonic function of exponential type less than π. If $u(n)=u(ne^{i\alpha})=0$ for all integers n, then u vanishes identically unless α is a rational multiple of π.*

The condition on α has also been of significance in the research of radial functions [1,9]. That it is necessary in the present theorem can be seen as follows. Given $\alpha=k\pi/l$, where $k, l\in\mathbb{Z}$, $l>0$, we consider the function u_l defined by

$$u_l(z) := \Im(z^l). \tag{1}$$

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(As usual, $\mathfrak{I}(z)$ denotes the imaginary part of z.) Then u_l is a real-valued entire harmonic function of exponential type zero which vanishes on the two lines $\{x: x \in \mathbb{R}\}$ and $\{xe^{i\alpha}: x \in \mathbb{R}\}$ but is not identically zero. However, a uniqueness theorem analogous to Theorem A does hold for $\alpha = \pi/2$ if the function u is supposed to be odd, i.e., if $u(-z) = -u(z)$ for all complex numbers z. Note that the function u_l defined in (1) is even for even l. Ching [6] proved the following statement.

THEOREM B. *Let u be a real-valued odd entire harmonic function of exponential type less than π. If $u(n) = u(ni) = 0$ for all integers n, then u vanishes identically.*

In the situation of Theorem B, Ching [6] also reconstructed u from its values at the lattice points n and ni, $n = 0, \pm 1, \pm 2, ...$, provided that u satisfies a certain growth condition.

THEOREM C. *Let u be a real-valued odd entire harmonic function of exponential type π such that the series $\sum_{n=-\infty}^{\infty} |u(n)|^p$ and $\sum_{n=-\infty}^{\infty} |u(ni)|^p$ are convergent, where $1 \leqslant p < \infty$. Then*

$$u(z) = \sum_{n=-\infty}^{\infty} u(n)\, v_n(z) + \sum_{n=-\infty}^{\infty} u(ni)\, v_n(-iz), \tag{2}$$

where

$$v_n(x+iy) := \frac{(-1)^n n[(x^2 - y^2 - n^2) \cosh \pi y \sin \pi x + 2xy \sinh \pi y \cos \pi x]}{\pi[y^2 + (x-n)^2][y^2 + (x+n)^2]} \tag{3}$$

and the series converge uniformly on every compact subset of $\mathbb{C}$.

Remark. The author [6] seems to have overlooked that, in case the exponential type of u is equal to π, the hypotheses of Theorem C

$$\sum_{n=-\infty}^{\infty} |u(n)|^p < \infty \qquad \text{and} \qquad \sum_{n=-\infty}^{\infty} |u(ni)|^p < \infty \tag{4}$$

are not strong enough. Indeed, the function $u(z) := \mathscr{R}(\sin \pi z)$ would satisfy (4) whereas the reconstruction formula (2) does obviously not hold. A correct version of Theorem C is obtained by replacing condition (4) by

$$u(\cdot) \in L^p(\mathbb{R}) \qquad \text{and} \qquad u(\cdot\, i) \in L^p(\mathbb{R}). \tag{5}$$

In view of Theorems of Plancherel and Pólya [4, Theorem 6.7.15 and Corollary 10.6.6] condition (5) implies (4) whereas (4) implies (5) only under the additional assumption that the exponential type of u is less than π.

Inspired by the example (1), we shall show that the supplement to Theorem A provided by Theorem B in case of $\alpha=\pi/2$ is also true whenever $\alpha=(2k+1)\,\pi/(2l)$ with integers k, l $(l\neq 0)$. Moreover, we establish a corresponding extension of Theorem C.

THEOREM 1. *Let u be a real-valued odd entire harmonic function of exponential type less than π. Let $\alpha=(2k+1)\,\pi/(2l)$ for some integers k and $l\neq 0$. If $u(n)=u(ne^{i\alpha})=0$ for all integers n, then u vanishes identically.*

THEOREM 2. *Let u be a real-valued odd entire harmonic function of exponential type π. Let $\alpha=(2k+1)\,\pi/(2l)$ for some integers k and $l\neq 0$ such that $u(\cdot)$, $u(\cdot\, e^{i\alpha})\in L^p(\mathbb{R})$ for some $p\in[1,\infty)$.*
Then

$$u(z)=\sum_{n=-\infty}^{\infty} u(n)\, w_n(-ze^{-i\alpha})+\sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\, w_n(z),$$

where

$$w_n(z):=\Im\left(\sum_{m=0}^{\infty}\frac{b_{2m+1}^{(n)}}{\sin((2m+1)\,\alpha)}\, z^{2m+1}\right) \tag{6}$$

and the numbers $\{b_m^{(n)}\}_{m\in\mathbb{N}}$ are uniquely defined by

$$\sum_{m=0}^{\infty} b_m^{(n)} z^m := \frac{\sin\pi(z-n)}{2\pi(z-n)}-\frac{\sin\pi(z+n)}{2\pi(z+n)}$$

$(n\in\mathbb{Z})$. *The series converge uniformly on every compact subset of $\mathbb{C}$.*

Our modification of Ching's method is also applicable if α/π is not rational but algebraic.

THEOREM 3. *Let u be a real-valued entire harmonic function of exponential type π satisfying $u(0)=0$. Let α/π be an irrational algebraic number. If $u(\cdot)$, $u(\cdot\, e^{i\alpha})\in L^p(\mathbb{R})$ for some $p\in[1,\infty)$, then*

$$u(z)=\sum_{n=-\infty}^{\infty} u(n)\,(-w_n(ze^{-i\alpha}))+\sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\, w_n(z),$$

where

$$w_n(z) := \Im\left(\sum_{m=1}^{\infty} \frac{b_m^{(n)}}{\sin(m\alpha)} z^m\right)$$

and the numbers $\{b_m^{(n)}\}_{m\in\mathbb{N}}$ *are uniquely defined by*

$$\sum_{m=0}^{\infty} b_m^{(n)} z^m := \frac{\sin \pi(z-n)}{\pi(z-n)}$$

$(n \in \mathbb{Z})$. *The series converge uniformly on every compact subset of* $\mathbb{C}$.

2. PROOFS OF THEOREMS

We shall need the following lemma.

LEMMA. *Let* u *be a real-valued entire harmonic function satisfying* $u(x) = u(xe^{i\alpha}) = 0$ *for all real numbers* x. *If*

$\alpha/\pi = (2k+1)/(2l)$ *and* u *is odd, or*

α/π *is irrational,*

then u *vanishes identically.*

Proof. Let v be a harmonic function conjugate to u so that $f := u + iv$ is entire. Then the functions $g(z) := f(z) + \overline{f(\bar{z})}$ and $h(z) := f(ze^{i\alpha}) + \overline{f(\bar{z}e^{i\alpha})}$ are also entire and vanish on the real line. Therefore, they vanish identically and we can conclude that

$$f(ze^{2i\alpha}) = -\overline{f(\bar{z})} = f(z) \tag{7}$$

holds for all complex numbers z.

(I) Let $\alpha = (2k+1)\pi/(2l)$ and let u be an odd function. Then applying (7) l times, we obtain that

$$f(z) = f(ze^{i\pi(2k+1)/l}) = f(ze^{i\pi\, 2(2k+1)/l}) = \cdots = f(ze^{i\pi(2k+1)}) = f(-z).$$

We have deduced that f is even. Therefore, the function u, which is the real part of f, is also even. Since u was supposed to be odd, it has to vanish identically.

(II) Now we suppose that α/π is irrational. The entire function f can be represented as $f(z) = \sum_{m=0}^{\infty} a_m z^m$. Then using (7), we find that $a_m(1 - \exp(2im\alpha)) = 0$ for all non-negative integers m. Since α/π is irrational, we can conclude that $a_m = 0$ for all $m > 0$, which implies that f has to be a constant. Therefore, u has to vanish identically. This completes the proof of the lemma. ▮

Proof of Theorem 1. Let v be a harmonic function conjugate to u so that $f := u + iv$ is entire. It follows from Carathéodory's inequality [4, Theorem 1.3.1] that f is of exponential type less than π (see also [3, proof of Theorem 1]). Therefore, the functions $g(z) := f(z) + \overline{f(\bar{z})}$ and $h(z) := f(ze^{i\alpha}) + \overline{f(\bar{z}e^{i\alpha})}$ are also entire and of exponential type less than π. Since we have $2u(x) = g(x)$ and $2u(xe^{i\alpha}) = h(x)$ for all real numbers x, they vanish at the integers. Hence by Carlson's theorem [4, Theorem 9.2.1], they vanish identically. In particular, $u(x) = u(xe^{i\alpha}) = 0$ for all real numbers x. Now the lemma yields that u itself vanishes identically. ▮

Proof of Theorem 2. Let us define

$$S_n(z) := \frac{\sin \pi(z-n)}{2\pi(z-n)} - \frac{\sin \pi(z+n)}{2\pi(z+n)}$$

for all integers n. Then S_n is an odd entire function. Therefore, S_n can be represented as $S_n(z) = \sum_{m=0}^{\infty} b_{2m+1}^{(n)} z^{2m+1}$. The sequence $\{\sin((2m+1)\,\alpha)\}_{m \geqslant 0}$ is periodic since α is a rational multiple of π. Furthermore, $\sin((2m+1)\,\alpha) \neq 0$ for all integers m. Therefore, the sequence $\{|\sin((2m+1)\,\alpha)|\}_{m \geqslant 0}$ is bounded from below by a positive real number and the function W_n defined by

$$W_n(z) := \sum_{m=0}^{\infty} \frac{b_{2m+1}^{(n)}}{\sin((2m+1)\alpha)}\, z^{2m+1}$$

$(n \in \mathbb{Z})$ is entire. As in the theorem, we define $w_n := \Im(W_n)$ for all integers n. Obviously, w_n is a real-valued entire harmonic function. Moreover, w_n is odd since W_n is odd. We shall need some more properties of w_n. It can be easily seen that the Taylor coefficients $b_m^{(n)}$ are real. Hence, W_n is real-valued on the real line and we get that

$$w_n(x) = 0$$

for all real numbers x. Furthermore, we have

$$\begin{aligned} w_n(xe^{\pm i\alpha}) &= \Im\left(\sum_{m=0}^{\infty} \frac{b_{2m+1}^{(n)}}{\sin((2m+1)\alpha)}\, e^{\pm i(2m+1)\alpha}\, x^{2m+1} \right) \\ &= \pm \sum_{m=0}^{\infty} b_{2m+1}^{(n)} x^{2m+1} \\ &= \pm S_n(x). \end{aligned} \tag{8}$$

Finally, we claim that

$$w_n(z) = O\left(\frac{1}{|n|}\right) \qquad \text{as} \qquad n \to \pm\infty \tag{9}$$

uniformly if z lies in a compact subset of $\mathbb{C}$. For a proof we recall [12, p. 11] that for two power series $F(z) = \sum_{m=0}^{\infty} c_m z^m$ and $G(z) = \sum_{m=0}^{\infty} d_m z^m$ the Hadamard product $F * G$ is defined by

$$(F * G)(z) := \sum_{m=0}^{\infty} c_m d_m z^m.$$

If F is analytic in $\{z \in \mathbb{C}: |z| < R\}$ for some $R > 0$ and if G is an entire function, then $F * G$ is also entire. A simple calculation yields the following representation (see also [12, p. 11])

$$(F * G)(z) = \frac{1}{2\pi i} \int_{|\zeta| = \varrho} F\left(\frac{z}{\zeta}\right) \frac{G(\zeta)}{\zeta}\, d\zeta \tag{10}$$

for $\varrho > |z|/R$. Let

$$F(z) := \sum_{m=0}^{\infty} \frac{z^{2m+1}}{\sin((2m+1)\,\alpha)},$$

which is analytic in $\{z \in \mathbb{C}: |z| < 1\}$. Then applying the formula (10) to the functions F and S_n, we obtain that

$$|(F * S_n)(z)| \leqslant \sup_{|\zeta| = \varrho} \left| F\left(\frac{z}{\zeta}\right) S_n(\zeta) \right| \leqslant \left(\frac{1}{|n| - \varrho}\right) \sup_{|\zeta| = \varrho} \left| F\left(\frac{z}{\zeta}\right) \frac{\sin \pi\zeta}{\pi} \right|$$

for $|z| < \varrho < |n|$. Note that $W_n = F * S_n$. Recalling that $w_n = \Im(W_n)$, we finally arrive at (9).

Let us define

$$w(z) := \sum_{n=-\infty}^{\infty} u(n)\, w_n(-ze^{-i\alpha}) + \sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\, w_n(z). \tag{11}$$

Applying Hölder's inequality, we get that

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} |u(n)\, w_n(-ze^{-i\alpha})| + \sum_{n=-\infty}^{\infty} |u(ne^{i\alpha})\, w_n(z)| \\ &\quad \leqslant \left(\sum_{n=-\infty}^{\infty} |u(n)|^p\right)^{1/p} \left(\sum_{n=-\infty}^{\infty} |w_n(-ze^{-i\alpha})|^q\right)^{1/q} \\ &\quad + \left(\sum_{n=-\infty}^{\infty} |u(ne^{i\alpha})|^p\right)^{1/p} \left(\sum_{n=-\infty}^{\infty} |w_n(z)|^q\right)^{1/q}, \end{aligned}$$

where $p \in [1, \infty)$ and $1/p + 1/q = 1$ (with obvious modifications for $p = 1$). Since u coincides with an entire function of exponential type on the real line and also coincides with another entire function of exponential type on the line $\{xe^{i\alpha}: x \in \mathbb{R}\}$ (see below), we obtain by using a result of Plancherel and Pólya [4, Theorem 6.7.15] that the series $\sum_{n=-\infty}^{\infty} |u(n)|^p$ and $\sum_{n=-\infty}^{\infty} |u(ne^{i\alpha})|^p$ are convergent. Therefore (9) guarantees that the series at the right hand side converge uniformly if z lies in a compact subset of $\mathbb{C}$. Thus, the series at the right hand side of (11) also converge uniformly on compact subsets of $\mathbb{C}$ and so w represents a real-valued entire harmonic function. Note that w is odd since w_n is odd for all integers n.

Let v be a harmonic function conjugate to u so that $f := u + iv$ is entire. Then f is of exponential type π as we have seen above, and so are the functions $g(z) := f(z) + \overline{f(\bar{z})}$ and $h(z) := f(ze^{i\alpha}) + \overline{f(\bar{z}e^{i\alpha})}$. Furthermore, they satisfy $g(x) = 2u(x)$ and $h(x) = 2u(xe^{i\alpha})$ for all real numbers x. Applying Shannon's sampling theorem [8, Theorem 1] and formula (8), we obtain that

$$
\begin{aligned}
u(x) &= \frac{1}{2} g(x) \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} g(n) \frac{\sin \pi(x-n)}{\pi(x-n)} \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} (u(n) - u(-n)) \frac{\sin \pi(x-n)}{\pi(x-n)} \\
&= \sum_{n=-\infty}^{\infty} u(n) \left(\frac{\sin \pi(x-n)}{2\pi(x-n)} - \frac{\sin \pi(x+n)}{2\pi(x+n)} \right) \\
&= \sum_{n=-\infty}^{\infty} u(n)\, w_n(-xe^{-i\alpha}) + \sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\; w_n(x) \\
&= w(x)
\end{aligned}
$$

for all real numbers x. Analogously, we find that

$$u(xe^{i\alpha}) = w(xe^{i\alpha})$$

for all real numbers x. This shows that the real-valued odd entire harmonic function $u - w$ vanishes on the two lines $\{x: x \in \mathbb{R}\}$ and $\{xe^{i\alpha}: x \in \mathbb{R}\}$. Hence by the lemma the function $u - w$ vanishes identically and Theorem 2 is proved. ∎

Proof of Theorem 3. Let us now define

$$S_n(z) := \frac{\sin \pi(z-n)}{\pi(z-n)}$$

for all integers n. The function S_n is entire and therefore it can be represented as $S_n(z) = \sum_{m=0}^{\infty} b_m^{(n)} z^m$. We now have to prove that for all integers n

$$W_n(z) := \sum_{m=1}^{\infty} \frac{b_m^{(n)}}{\sin(m\alpha)} z^m$$

is also an entire function. Note that $\sin(m\alpha) \neq 0$ for all positive integers m since α is not a rational multiple of π. Furthermore, we have [11]

$$|m\alpha - j\pi| > \frac{C}{m^2}$$

for a positive real number C and all integers $m > 0$ and j. This implies

$$\left|\frac{1}{\sin(m\alpha)}\right| \leqslant C'm^2$$

for a positive real number C' and all positive integers m. Therefore, we obtain

$$\limsup_{m\to\infty} \left|\frac{b_m^{(n)}}{\sin(m\alpha)}\right|^{1/m} \leqslant \limsup_{m\to\infty} |b_m^{(n)}|^{1/m} \limsup_{m\to\infty} (C'm^2)^{1/m} = 0.$$

This shows that W_n is entire. As in the theorem, we define $w_n := \Im(W_n)$ for all integers n. Then w_n is a real-valued entire harmonic function. As in the proof of Theorem 2, we can deduce the following properties of w_n;

$$w_n(x) = 0 \qquad \text{for} \quad x \in \mathbb{R};$$

$$w_n(xe^{\pm i\alpha}) = \pm S_n(x) \qquad \text{for} \quad n \neq 0 \quad \text{and} \quad x \in \mathbb{R};$$

$$w_n(z) = O\left(\frac{1}{|n|}\right) \qquad \text{as} \qquad n \to \pm\infty$$

uniformly if z lies in a compact subset of $\mathbb{C}$.

Let us define

$$w(z) := \sum_{n=-\infty}^{\infty} u(n)\,(-w_n(ze^{-i\alpha})) + \sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\, w_n(z).$$

Using the same arguments as in the proof of Theorem 2, we see that w represents a real-valued entire harmonic function.

Let v be a harmonic function conjugate to u so that $f := u + iv$ is entire. Then, f is of exponential type π. Therefore, the functions $g(z) := f(z) + \overline{f(\bar{z})}$ and $h(z) := f(ze^{i\alpha}) + \overline{f(\bar{z}e^{i\alpha})}$ are also entire and of exponential type π. Furthermore, they satisfy $g(x) = 2u(x)$ and $h(x) = 2u(xe^{i\alpha})$ for all real numbers x. Applying Shannon's sampling theorem again, we obtain

$$\begin{aligned} u(x) &= \frac{1}{2} g(x) \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} g(n) \frac{\sin \pi(x-n)}{\pi(x-n)} \\ &= \sum_{n=-\infty}^{\infty} u(n)\,(-w_n(xe^{-i\alpha})) + \sum_{n=-\infty}^{\infty} u(ne^{i\alpha})\, w_n(x) \\ &= w(x) \end{aligned}$$

for all real numbers x. Note that $u(0) = 0$. Analogously, we get that

$$u(xe^{i\alpha}) = w(xe^{i\alpha})$$

for all real numbers x. Applying the lemma to the real-valued entire harmonic function $u - w$, we find that $u - w$ vanishes identically. Thus, Theorem 3 is proved. ∎

3. FINAL REMARKS

In case of $\alpha = \pi/2$ it can be easily seen that the odd entire harmonic functions v_n and w_n defined by (3) and (6), respectively satisfy

$$v_n(x) - w_n(ix) = v_n(ix) - w_n(-x) = 0$$

for all real numbers x. Applying the lemma to the function $v_n(\cdot) - w_n(\cdot\, i)$, we find that $v_n(\cdot)$ and $w_n(\cdot\, i)$ are identical. Hence, Theorem 2 is an extension of Theorem C.

These results can be seen as a partial answer to a conjecture raised by Ching in [6]. Ching conjectured that for general $\alpha = k\pi/l$ there exists an interpolation formula analogous to (2) for a certain class of harmonic functions for which a uniqueness theorem holds. We proved that this is the case if $\alpha = (2k+1)\pi/(2l)$. In case of $\alpha = k\pi/(2l+1)$ we can find non-trivial even and odd entire harmonic functions of exponential type zero vanishing on the two lines $\{x: x \in \mathbb{R}\}$ and $\{xe^{i\alpha}: x \in \mathbb{R}\}$. Indeed, the functions $u_{2(2l+1)}$ and u_{2l+1} defined in (1) have the desired properties. Let us mention that there are many other entire harmonic functions which vanish on the two lines given above. Whenever $f(z)=\sum_{m=0}^{\infty} a_m z^m$ is an entire function of exponential type τ which is real-valued on the real line, then $u(z):=\Im(\sum_{m=1}^{\infty} a_{ml} z^{ml})$ is an entire harmonic function of exponential type τ vanishing on $\{x: x\in\mathbb{R}\}$ and $\{xe^{i\alpha}: x\in\mathbb{R}\}$ provided that $\alpha=k\pi/l$. If l is even, obviously u is also even. If l is not even, then we cannot say anything about the symmetry properties of u.

Finally, let us mention that by using similar methods as presented above we can also find reconstruction formulas for entire harmonic functions with non-uniform nodes $\{t_n^{(1)}\}_{n\in\mathbf{Z}}$ and $\{t_n^{(2)}e^{i\alpha}\}_{n\in\mathbf{Z}}$, where the real numbers $t_n^{(1)}$ and $t_n^{(2)}$ have to satisfy the condition $|t_n^{(j)}-n|\leqslant L$ $(j=1, 2)$ for a suitable L.

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